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Math 12 Honours: section 5.5 Applications of Logarithms and Exponential Functions:

1. Suppose you invested \$1000 at 5% compounded annually, how many years will it take your investment to grow to i) \$5000? Ii) \$10,000 iii) \$100,000

i) $1000(1.05)^t = 5000$

$t = \log_{1.05} 5 = \boxed{32.99}$

ii) $1000(1.05)^t = 10000$

$t = \log_{1.05} 10 = \boxed{47.2}$

iii) $t = \log_{1.05} 100 = \boxed{94.4}$

2. Calculate the number of years it would take an investment to double for each of the following:

i) 10% compounded semi-annually

$2\cancel{P} = \cancel{P}\left(1+\frac{0.1}{2}\right)^{2n}$

$2n = \log_{1.05} 2 \Rightarrow \boxed{n = 7.10}$

ii) 10% compounded daily

$2\cancel{P} = \cancel{P}\left(1+\frac{0.1}{365}\right)^{365n}$

$365n = \log_{\left(1+\frac{0.1}{365}\right)} 2 \Rightarrow \boxed{n = 6.93}$

iii) 8% compounded weekly

$2\cancel{P} = \cancel{P}\left(1+\frac{0.08}{52}\right)^{52n} \Rightarrow 52n = \log_{\left(1+\frac{0.08}{52}\right)} 2$

$\boxed{n = 8.67}$

iv) 8% compounded monthly

$2P = P\left(1+\frac{0.08}{12}\right)^{12n} \Rightarrow 12n = \log_{\left(1+\frac{0.08}{12}\right)} 2$

$\boxed{n = 8.69}$

3. In general, for "k%" interest, how many years does an investment take to double? Triple?

$A = P \cdot e^{r \cdot t}$

Double

$2\cancel{P} = \cancel{P}e^{kt}$

$kt = \ln 2 \Rightarrow \boxed{t = \frac{0.6931472}{k}}$

Triple

$t = \frac{\ln 3}{k} \Rightarrow \boxed{t = \frac{1.0986123}{k}}$

4. A 5% investment compounded daily is equivalent to what interest rate compounded annually?

$\cancel{P}\left(1+\frac{0.05}{365}\right)^{365n} = \cancel{P}(1+k)^n$

$\left(1+\frac{0.05}{365}\right)^{365} - 1 = k \Rightarrow \boxed{k = 0.051267}$

It's equivalent to 5.1267% interest

5. Suppose you invest \$5000 each year into an investment that gives a 10% return, how much will your portfolio be after 25 years?

$\left\{\left[(5000 \times 1.1) + 5000\right] \times 1.1 + 5000\right\} \times 1.1 \quad \ldots$

$ar^n + ar^{n-1} + \cdots + a = \frac{ar^n - a}{r-1}$

$5000 \times 1.1^{25} + 5000 \times 1.1^{24} + \cdots + 5000 \times 1.1 + 5000$

$= \frac{5000 \times 1.1^{26} - 5000 \times 1.1}{1.1 - 1} = \boxed{540,908.83}$

6. The half life of sodium-24 is 15.5 hours. Suppose a hospital buys a 100mg sample of sodium-24, how much of the sample will be left after 72 hours?

$F = I \times N^{\frac{A}{L}}$

$F = 100 \times \left(\frac{1}{2}\right)^{\frac{72}{15.5}} = \boxed{3.996mg}$

F = final amount left
I = initial amount
N = rate of decay/multipication
A = total time passed
L = time required for rate to happen.

7. How long will it take a 100mg sample of sodium-24 to decompose to 1mg?

$F = I \times N^{\frac{A}{L}}$

$1 = 100 \times \left(\frac{1}{2}\right)^{\frac{A}{15.5}} \Rightarrow \frac{A}{15.5} = \log_{0.5} 0.01 \Rightarrow \boxed{A = 102.98 \text{ hours}}$

8. Iodine-129 is a dangerous substance in radioactive waste with a half life of 16million years. How long will it take 1,000g of Iondine-129 to decompose to 1g?

$F = I \times N^{\frac{A}{L}}$

$1 = 1000 \cdot \left(\frac{1}{2}\right)^{\frac{A}{16}} \Rightarrow A = 16 \log_{0.5} 0.001 = \boxed{159.45 \text{ million years}}$

9. A 500mg radioactive substance decomposed to 20mg in 25 years. What is the half life of the radioactive substance?

$F = I \times N^{\frac{A}{L}}$

$20 = 500 \cdot \left(\frac{1}{2}\right)^{\frac{25}{L}} \Rightarrow \frac{25}{L} = \log_{0.5}\left(\frac{20}{500}\right) \Rightarrow \boxed{L = 5.38 \text{ years}}$

10. A large fossil with 25mg of carbon 14 was found that originally contained 3000mg of carbon 14. If the half life of carbon 14 is 5700 years, then how old is the fossil?

$25 = 3000 \left(\frac{1}{2}\right)^{\frac{A}{5700}} \Rightarrow A = 5700 \log_{0.5}\left(\frac{25}{3000}\right) \Rightarrow \boxed{A = 39{,}369.28 \text{ years}}$

11. 5% of a substance decays in 100 days. What is the half life of the substance?

$0.95\cancel{x} = \cancel{x}\left(\frac{1}{2}\right)^{\frac{100}{L}} \Rightarrow \frac{100}{L} = \log_{0.5} 0.95 \Rightarrow \boxed{L = 1351.34 \text{ days} = 3.7 \text{ years}}$

If 5% decays in 100 years, you'd have 95% of it left.

12. A swarm of locusts can multiply in population by 5 times every 6 weeks. How long will it take a cluster of 5000 locusts to grow to 1 million?

$$1000000 = 5000(5)^{\frac{A}{6}} \Rightarrow A = 6\log_5 200 = \boxed{19.75 \text{ weeks}}$$

13. The amount of intensity of an earthquake is measured by the amount of ground motion recorded on a seismograph. An increase in 1unit magnitude on the Ritcher scale represents a 10 fold in intensity. How many times more 'intense' is an earthquake of magnitude 9 than an earthquake of 5.5?

$$I_1 = I_2\left(10^{R_1-R_2}\right)$$

It is 31622.78 times more intense

$$\frac{I_1}{I_2} = 10^{9-5.5} \Rightarrow \frac{I_1}{I_2} = \boxed{31622.78}$$

14. For each increase of 1 unit in magnitude, earthquakes are 32 times more powerful in the amount of energy. An earthquake of magnitude of 1 has about $794,328J$ of energy. An earthquake of magnitude 2 has about $25,118,864J$ of energy. An earthquake of 2.5 magnitude has 141,253,754J of energy. How many much energy is in an earthquake with an magnitude of 6.5?

$$E_2 = E_1\left(32^{6.5-1}\right)$$

E_2 = magnitude 6.5

E_1 = magnitude 1

$$E_2 = 794,328\text{J}\left(32^{5.5}\right) = \boxed{1.5077 \times 10^{14} \text{ J}}$$

15. In 1976, an earthquake in Italy released approximately 10^{14} joules of energy. What is the magnitude of that earthquake?

$$E = E_1\left(32^{m-1}\right) \Rightarrow m-1 = \log_{32}\left(\frac{10^{14}\text{ J}}{794,328\text{J}}\right) \Rightarrow \boxed{6.3815}$$

16. The loudness of sound is measured in decibals (db). Every increase in 10db represents a 10 fold increase in loudness. Going from 10db to 30db, an increase in 20db, is 10^2 times more loud. A regular conversation is about 55db. Listening to a Rock concert is about 110db. How many times louder is the Rock concert than a normal conversation?

$$I_f = I_i \cdot 10^{\frac{d_f - d_i}{10}} \Rightarrow \frac{I_f}{I_i} = 10^{\frac{110-55}{10}} = \boxed{316227.77}$$

17. A soft whisperer speaks at about 30db. Getting in a yelling match with Cheong is about a thousand times louder. How many db would Cheong be shouting at?

$$I_f = I_i \cdot 10^{\frac{d_f - d_i}{10}} \Rightarrow \frac{I_f}{I_i} = 1000 = 10^{\frac{d_f - 30}{10}} \Rightarrow \frac{d_f - 30}{10} = 3 \Rightarrow \boxed{d_f = 60 \text{ db}}$$

18. In chemistry, the pH scale measures acidity or alkalinity of a solution. pH ranges from 0 (acidic) to 14 (alkalinity), with 7 being neutral. For each increase in pH, there is a 10 fold increase in alkalinity. For each decrease in pH, there is a 10 fold increase in acidity. Tomato juice has a pH of 4.2 and lemon juice is about 1.8 in pH. How much more acidic is lemon juice than tomato juice?

$$I_2 = I_1 \cdot 10^{P_2 - P_1} \Rightarrow \frac{I_2}{I_1} = 10^{4.2-1.8} = \boxed{251.189}$$

19. A dishwasher soap has a pH of 12.8. Baking soda is about 9.3 in pH. How much more alkaline is the dishwater soap than baking soda?

$$I_2 = I_1 \cdot 10^{P_2 - P_1} \Rightarrow \frac{I_2}{I_1} = 10^{12.8-9.3} = \boxed{3162.28}$$

20. The pH values of milk ranges from 6.4 to 7.6. Pure water has a pH value of 7.0. How much more acidic is milk with a pH of 6.4 than pure water?

$$\frac{I_2}{I_1} = 10^{7-6.4} = \boxed{3.98}$$

21. How much more alkaline is milk with a pH of 7.6 than pure water?

$$\frac{I_2}{I_1} = 10^{7.6-7} = \boxed{3.98}$$

22. The total amount of arable land in th world is about $3.2 \times 10^9\, ha$. A ~~n~~ hectare of land is about 100m by 100m. 0.4 ha of land is enough to grow food for one person. World population in 2021 is 7.874billion, growing at 1.1% annually. When will the demand for arable land exceed the supply available?

$$7.874 \cdot 10^9 (1.011)^n \times 0.4 = 3.2 \times 10^9 \Rightarrow n = \log_{1.011} 1.016 = \boxed{1.451}$$

In 1.451 years

23. If we are able to reduce the growth rate of the world population by 0.5 and increase the productivity of food production by 200%, then how long will we have until the demand for arable land exceed supply?

x3 original

growth rate $= \frac{1.011}{2} = 1.0055$

land per person $= \frac{0.4}{3}$ ha

$$7.874 \times 10^9 (1.0055)^n \cdot \frac{0.4}{3} = 3.2 \times 10^9$$

$$n = \log_{1.0055} 3.04801 = \boxed{203.2 \text{ years}}$$